

F-theory, Geometric Engineering and N=1 Dualities

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We consider geometric engineering of $N = 1$ supersymmetric QFTs with matter in terms of a local model for compactification of F-theory on Calabi-Yau fourfold. By brining 3-branes near 7-branes we engineer $N = 1$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors in the fundamental. We identify the Higgs branch of this system with the instanton moduli space on the compact four dimensional space of the 7-brane worldvolume. Moreover we show that the Euclidean 3-branes wrapped around the compact part of the 7-brane worldvolume can generate superpotential for $N_f = N_c - 1$ as well as lead to quantum corrections to the moduli space for $N_f = N_c$. Finally we argue that Seiberg's duality for $N = 1$ supersymmetric QCD may be mapped to T-duality exchanging 7-branes with 3-branes.

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1. Geometric Engineering of SUSY QCD

Many non-trivial aspects of field theory dualities in four dimensional supersymmetric systems have found a natural interpretation in the context of string theory. This is the case in particular for $N=4$ and $N=2$ supersymmetric theories where, by a suitable engineering of a local model for the compactification manifold the non-trivial dualities of field theories can be reduced to an application of T-duality (see e.g. [1] for the $N = 2$ case). The case of $N=1$ theories has been the most difficult case to study, though there has been some partial progress. In this paper we find a natural geometric engineering of $N = 1$ quantum field theories with matter in four dimensions which again maps $N = 1$ dualities [2] to T-dualities of string theory.

The basic approach is via F-theory, where we consider compactification on elliptic Calabi-Yau 4-folds from 12 dimensions down to 4. This leads to $N = 1$ quantum field theories in four dimensions. In particular a situation with pure $N=1$ Yang-Mills in $d = 4$ was geometrically engineered in [3]. This was done by constructing a 7-brane over which the elliptic fibration has ADE singularity. The question we mainly concentrate in this paper is a way to incorporate matter into this system. One way of doing this is in principle to have extra singularities over the space as in the $N = 2$ case [4] [5]. There seems to be certain obstacles in this way of getting matter in the $N=1$ theories, because almost inescapably when one attempts to put extra singularities they are of the form which does not seem to correspond to conventional physics [6] (suggesting exotic physics as in [7]). Here we attempt a different way of incorporating matter into the system.

As was noted in [8] the F-theory on Calabi-Yau 4-fold induces a certain number of 3-brane charge which needs to be cancelled in order to get a consistent theory. One way of doing this is by adding a certain number of 3-branes which fill the spacetime. The number of 3-branes needed for this is $\chi/24$ where χ is the Euler characteristic of Calabi-Yau fourfold (see also [9]). Now suppose we have a 7-brane which has pure ADE singularity, without extra singularities. Note that the 7-brane worldvolume is $R^4 \times S$ where S is a complex 2 dimensional surface in the base of the elliptic 4-fold. In general, in addition to $N = 1$ Yang-Mills, we get $h^{2,0}(S) + h^{1,0}(S)$ adjoint chiral fields [3]. We will first consider the case where we do not have any adjoint fields and comment on generalization for the case with adjoints later in the paper. This means that we first consider the case

$$h^{1,0}(S) = h^{2,0}(S) = 0 \tag{1.1}$$

This assumption, together with the simplifying assumption of rigidity of S implies that S is a rational surface. This local model leads to pure ADE Yang-Mills in $d = 4$ with no matter. Let us concentrate on the case of A_{N_c-1} singularity which gives rise to $SU(N_c)$ Yang-Mills. In order to obtain pure field theory results (i.e. turning off gravitational/stringy effects) we need to consider the limit where the volume of S is very large. Note that the position of the 3-branes which fill the spacetime can vary over the three dimensional base. Suppose we bring one of the 3-branes near the 2-dimensional surface S . In this way we can analyze the result by a local analysis, which leads to having a hypermultiplet in the fundamental of $SU(N_c)$ (which is also charged under the $U(1)$ charge on the 3-brane) [10]. In terms of $N = 1$ matter, this corresponds to quark fields Q and \tilde{Q} in the representations \mathbf{N}_c and $\overline{\mathbf{N}}_c$. Moreover out of the 3 chiral fields which correspond to moving the 3-brane around, one of them, which takes the 3-brane off of S , can give mass to the quark field. If we bring N_f of the 3-branes near S we obtain an $N = 1$ Yang-Mills theory with gauge group $SU(N_c)$ and with N_f flavors Q, \tilde{Q} .

This system has two branches [11]. On the one hand we can give mass to the quarks, which as noted above corresponds to taking the 3-branes off the 7-brane surface S . There is also a Higgs branch. Note that if $N_f < N_c - 1$ maximal Higgsing will leave us with $SU(N_c - N_f)$ unbroken gauge symmetry with no matter. On the other hand when $N_f \geq N_c - 1$ we can higgs the group completely. The classical moduli space of the Higgs branch is obtained by considering the gauge invariant observable made out of Q, \tilde{Q} . The mathematical way of saying this is that the moduli space of Higgs branch is simply

$$\mathcal{M} = \frac{\mathbf{C}^{2N_f N_c}}{SL(N_c)} \quad (1.2)$$

where the complex space $\mathbf{C}^{2N_f N_c}$ corresponds to expectation values for Q, \tilde{Q} and the quotient represents the gauge invariant observables (see e.g. [12]). For $N_f > N_c$ the complex structure of this moduli space does not receive quantum corrections whereas for $N_f = N_c$ it receives quantum corrections [13]. Note that the complex dimension of the moduli space when complete Higgsing is possible is given by

$$\dim \mathcal{M}^{Higgs} = 2N_f N_c - (N_c^2 - 1) \quad (1.3)$$

The question is how this branch of the moduli space is realized in the context of F-theory under discussion. The answer is very simple, once we note that 3-branes within

7-branes can be equivalently viewed as corresponding to zero size instantons of the corresponding 7-brane [14] [15] [16]. If we consider finite size instantons, we break the $SU(N_c)$ gauge symmetry and will thus Higgs the system¹. Thus *we identify* the Higgs branch of this system with the moduli of $SU(N_c)$ instantons on S with instanton number N_f . As a first check let us see what the dimension of this moduli space is. For any gauge group G on a four dimensional space S with instanton number k the complex dimension of moduli space is (assuming instantons exist)

$$\dim \mathcal{M}^{inst} = 2kc_2(G) - \dim(G) \frac{[\chi + \sigma]}{4} = 2kc_2(G) - \dim(G)(h^{2,0} - h^{1,0} + h^{0,0}) \quad (1.4)$$

where $c_2(G)$ denotes the dual coxeter number of the group G and χ and σ denote the Euler characteristic and the signature of S respectively whose sum is related to the hodge numbers of S as indicated above. For the case at hand $k = N_f$ and $c_2(SU(N_c)) = N_c$, and we are taking $h^{1,0} = h^{2,0} = 0$. Thus we find that the dimension of instanton moduli space is

$$\dim \mathcal{M}^{inst.} = 2N_f N_c - (N_c^2 - 1)$$

in agreement with (1.3). This formula is valid for $N_f \geq N_c - 1$ (there is some subtlety for $N_f = N_c - 1$ noted below). Not only the dimensions match, but one can argue that in fact the moduli space is identical to the expectation based on the gauge theory realization (1.2).

This comes from a well known mathematical construction [18] for instantons on rational surfaces which we now review. It can be shown that $SU(N_c)$ instantons on S exist if and only if the instanton number N_f satisfies $N_f \geq N_c$. (For the case of $N_f = N_c - 1$ we can only define an $SU(N_c - 1)$ instanton. This special case will be treated later.) Note that these facts are in accord with the cases of complete Higgsing expected from the field theory analysis. Now let us consider the detailed construction. We will always be interested in a piece of the moduli of instantons on a rational surface which is a neighborhood of an instanton of zero size. Geometrically this amounts to looking at the restriction of instantons on the complement of finitely many curves on the surface. Since all rational surfaces become isomorphic after we throw out some curves we have the freedom to choose a particular model of our surface. For simplicity we concentrate on the case where S is a

¹ Aspects of this transition has been considered recently in connection with $N = 1$ F-theory/heterotic dualities in [17].

Hirzebruch surface. For example we can take $S = \mathbf{P}^1 \times \mathbf{P}^1$. More generally we consider S to be a \mathbf{P}^1 bundle over \mathbf{P}^1 given by $\pi : S \rightarrow \mathbf{P}^1$. Let us denote the moduli space of vector bundles of rank N_c with vanishing first Chern class and second Chern class equal to N_f by $\mathcal{M}_S(N_c, 0, N_f)$. Below we will describe an open dense subset in $\mathcal{M}_S(N_c, 0, N_f)$. First we fix a special vector bundle W over S of rank N_c and zero instanton number, $c_2(W) = 0$. The instanton number can be changed by a procedure known as the *Hecke transform* which we will describe below. We consider a divisor $D = \sum_1^{N_f} y_i$ consisting of N_f distinct \mathbf{P}^1 fibers of π . A generic vector bundle is realized as a Hecke transform of W along divisor D . Each Hecke transform will increase the instanton number by 1 and can roughly be associated with giving a zero size instanton a finite size. Since these Hecke transforms occur along the N_f fibers the resulting bundle will have second Chern class equal to N_f . All moduli of the vector bundles are encoded in the deformations of the Hecke transform.

We briefly review the definition of a Hecke transform. The Hecke transform of a vector bundle $W \rightarrow S$ along a divisor $D \subset S$ depends on the additional choice of a vector bundle F on D and a surjective map of vector bundles $\xi : W|_D \rightarrow F$. This additional data is called a center of the Hecke transform. Given W , F and ξ define the Hecke transformed bundle \widetilde{W} of W by the exact sequence

$$0 \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow F \longrightarrow 0, \quad (1.5)$$

where the map $W \rightarrow F$ is given by ξ . It turns out that \widetilde{W} is a locally free sheaf of the same rank as W . Moreover \widetilde{W} coincides with W everywhere except along the divisor D . Sometimes the Hecke transform is called an elementary modification of W . For more details see [19].

Now let us come back to the situation at hand. We first wish to define the bundle W . Any rank N_c bundle over \mathbf{P}^1 can be written as a sum of line bundles. Consider the integers a and b defined by $N_f = aN_c + b$ where $0 \leq b < N_c$. We define the bundle W over S to be the pull-back

$$W = \pi^* \left(\mathcal{O}_{\mathbf{P}^1}(a)^{\oplus(N_c-b)} \oplus \mathcal{O}_{\mathbf{P}^1}(a+1)^{\oplus b} \right).$$

The divisor D is a collection of N_f copies of \mathbf{P}^1 and we choose F to be the line bundle $\mathcal{O}(1)$ on each of these \mathbf{P}^1 's. Note that $\mathcal{O}(1)$ has two sections. The restriction $W|_D$ is the trivial rank N_c vector bundle on D . In order to specify a Hecke transform of W we need also a surjective map $\xi : W|_D \rightarrow F$. The moduli of all such ξ 's are

$$\text{Hom}(W|_D, F) = \bigoplus_{i=1}^{N_f} \text{Hom}(\mathcal{O}_{y_i}^{\oplus N_c}, \mathcal{O}_{y_i}(1)) = \bigoplus_{i=1}^{N_f} H^0(y_i, \mathcal{O}_{y_i}(1)^{\oplus N_c}). \quad (1.6)$$

Since each fiber y_i is a \mathbf{P}^1 and the global sections $H^0(\mathbf{P}^1, \mathcal{O}(1))$ are just the two homogeneous coordinates, the right hand side of (1.6) is simply a complex vector space of dimension $2N_c N_f$ and we identify $\xi = (Q, \tilde{Q})$ squark fields, where each squark field goes into defining a Hecke transform on each component y_i of the divisor D . Two different maps ξ and ξ' lead to the same vector bundle whenever they differ by automorphisms of the bundle W and the sheaf F . The inverse is also true [18]. Since the k -tuples of points on \mathbf{P}^1 are parametrized by a N_f -dimensional vector space that can be thought of as a quotient of $\mathbf{C}^{N_f+1} - \{0\}$ by \mathbf{C}^* we can describe the full instanton moduli $\mathcal{M}_S(N_c, 0, N_f)$ as a quotient of an open set in $\mathbf{C}^{2N_c N_f} \oplus \mathbf{C}^{N_f}$ by the group $Aut(W)/\mathbf{C}^* \times Aut(F) \times \mathbf{C}^*$. This description of moduli space is dual to the one given in [18]. The automorphism group of F is \mathbf{C}^{*k} . Since we are interested in the limit of the moduli space when the size of S is growing to infinity we need the corresponding limit of this quotient. But in this limit the base \mathbf{P}^1 goes to infinite size and hence the construction will remember only the restriction of W on the complement of a fiber of π and hence $Aut(W)/\mathbf{C}^*$ reduces to $SL(N_c, \mathbf{C})$. In particular, the actions of $Aut(W)/\mathbf{C}^*$ and \mathbf{C}^{*k} decouple and in the limit the moduli looks like $\mathbf{C}^{2N_c N_f}/SL(N_c)$.

2. Quantum Moduli Space

So far we have only discussed how the classical aspects of the gauge system can be engineered in the context of F-theory compactification on 4-folds. As was argued in [13] the classical moduli space receives a quantum correction only for $N_f = N_c$, where if we consider the $N_f \times N_f$ meson field $M = Q\tilde{Q}$ and let $B = \det Q$ and $\tilde{B} = \det \tilde{Q}$ denote the two baryon fields, then the quantum moduli space is

$$\det M - B\tilde{B} = \Lambda^{2N_c}$$

where $\Lambda^{2N_c} = \mu^{2N_c} \exp(\frac{-8\pi^2}{g^2})$ comes from a point-like instanton configuration. Moreover for the case $N_f = N_c - 1$ one expects a superpotential due to point-like instantons [20]

$$W = \frac{\tilde{\Lambda}^{2N_c+1}}{\det M}$$

We now wish to look for such corrections in the present setup. The basic instanton to consider can be geometrically understood in the present context by noting that a point like instanton in uncompactified spacetime for the 7-brane, corresponds to a Euclidean

3-brane wrapping around S , whose action goes as $\sim \exp(-V)$ where V is the volume of S . Moreover the volume of S is indeed $\frac{1}{g^2}$ (from the reduction of the gauge theory from 8 dimensions down to 4 on S). This is thus in agreement with the form of the correction. To argue that there is such a correction, we will have to do the zero mode analysis for this euclidean three brane similar to what was done for Euclidean membranes [21] and euclidean fivebranes [22]. The analysis relevant for us is a simple extension of [22]. The main new novelty here is that when a Euclidean 3-brane wraps S this will lead to a hypermultiplet in the fundamental of $SU(N_c)$ corresponding to the open string stretched between the Euclidean 3-brane and the (N_c) 7-branes. This mode propagates on the euclidean worldvolume S of the 3-brane. The reader should be careful to distinguish between the N_f fundamentals of $SU(N_c)$ which propagate on uncompactified spacetime and the one fundamental of $SU(N_c)$ which lives on S . We consider a particular point on the Higgs moduli, which corresponds to some fixed $SU(N_c)$ instanton on S , with instanton number N_f (i.e. giving vevs to the spacetime squarks). The hypermultiplet living on the 3-brane worldvolume propagates on this fixed background. The Euclidean 3-brane worldvolume theory is twisted [23] and in this case the twisting is the one considered in [24]. This twisting leads to fermions in the hypermultiplet being represented by anti-holomorphic forms $\Omega^{0,0} \oplus \Omega^{0,1} \oplus \Omega^{0,2}$ with coefficients in the fundamental representation of $SU(N_c)$. Just as in [22] we can define a conserved charge W which basically corresponds to the twisting of the theory along S . If we consider the charge carried by the fermions it is $\pm \frac{1}{2}$ correlated with the degree of the form (i.e. the chirality of the spinor before twisting). The net violation of this charge due to fermion zero modes (which is doubled because of the taking into account both components of a hypermultiplet), is given by the index of $\bar{\partial}_A$ where A denotes the $SU(N_c)$ gauge connection on S . The index for this complex is found by a simple application of Atiyah-Singer index theorem to be given by

$$\Delta W = \text{ind}(\bar{\partial}_A) = n_0 - n_1 + n_2 = N_c - N_f \quad (2.1)$$

where n_i denotes the number of holomorphic section of degree i -antiholomorphic forms with coefficient in the fundamental representation of $SU(N_c)$. As argued in [22] in order for the instanton to contribute to the superpotential there must be a violation of $\Delta W = 1$ (corresponding to the charge of $d^2\theta$ and thus a contribution to the superpotential). In the same way one would expect that if we want a correction to the complex moduli of the vacua we should have zero net violation of W , i.e., $\Delta W = 0$ [25]. This would in particular lead

to a non-vanishing contribution from the instanton without involving the integration over $d^2\theta$. Since this is an instantonic contribution it should only affect holomorphic quantities and this should thus lead to quantum correction to the moduli space. From the formula (2.1) we see that the case corresponding to quantum correction to the moduli space is $N_f = N_c$ and the case corresponding to generation of superpotential is $N_f = N_c - 1$. As noted in the quantum field theory setup, the case $N_f = N_c$ is indeed expected to lead to quantum correction to the complex structure of moduli space, confirming the above analysis. Note also that this contribution is of the form $\exp(-V) = \exp(-1/g^2)$ given the relation between the volume of S and the gauge coupling constant. Also the case with $N_f = N_c - 1$ is the case where one expects point-like instanton contributions to the superpotential. However we should analyze this case more carefully, because our analysis for instantons applied only to the cases where $N_f \geq N_c$.

Let us now consider the $N_f = N_c - 1$ case in detail. In this case as mentioned before we cannot have $SU(N_c)$ instantons on S , but we can have $SU(N_c - 1)$ instantons on it. This will in effect describe the Higgs branch in this case. The reason for this is that giving vev to squarks in the $SU(N_c - 1)$ subgroup of the color group effectively Higgses the group completely. Let us see if this is also in agreement with the dimension of moduli of $SU(N_c - 1)$ instantons with instanton number $SU(N_c - 1)$. The dimension of instanton moduli space in this case is

$$\dim \mathcal{M}^{inst.} = (N_c - 1)^2 + 1$$

However taking into account that with an $SU(N_c - 1)$ instanton, we still have an $U(1)$ symmetry of the original $SU(N_c)$ theory left, this acts by a C^* action on the above moduli space leaving us with the reduced moduli space of dimension $(N_c - 1)^2$. This is easily verified to be the dimension of the Higgs branch expected in this case. Now let us return to the contribution of the Euclidean 3-brane instanton in this case. The 1 fundamental hypermultiplet propagating on S will be decomposed into a fundamental of $SU(N_c - 1)$ plus a singlet. The net violation of W will come only from the singlet component, in which case

$$\Delta W = n_0 - n_1 + n_2 = h^{0,0} - h^{0,1} + h^{0,2} = 1$$

We will thus get a correction to the superpotential as expected. To find what the superpotential contribution of the Euclidean 3-brane is, in addition to the instanton action which leads to the $\exp(-1/g^2)$ we will get contribution from the determinant of non-zero

modes of the fields in the instanton background [22]. In this case we have to also take into account the bosonic components of the hypermultiplet, which belong to degree 0 and 2 anti-holomorphic forms coupled to the fundamental representation of $SU(N_c - 1)$. Putting the contribution of non-zero modes for bosons and fermions one finds that the prefactor is precisely the Ray-Singer torsion² of the $\bar{\partial}_A$, which is given by

$$RS = \prod_p \det \Delta_p^{-(-1)^p p}$$

where Δ_p denotes the laplacian acting on anti-holomorphic p-forms with coefficient in the fundamental of $SU(N_c - 1)$. The Ray-Singer torsion RS will depend on the complex moduli of the instanton. Given the fact that the instanton moduli space is $SU(N_f)$ symmetric, this can only be a function of $\psi = \det M$. Now consider the limit where we turn off one of the instantons. This means that we consider a point where M has one lower rank than the maximal rank of $N_f = N_c - 1$. In other words this corresponds to the point where $\psi = 0$. In this case we find that there is an extra bosonic zero mode contribution to the Ray-Singer torsion. This follows from the fact that the index of $\bar{\partial}_A$ goes up by one when we shrink one instanton and the RS torsion has the same order of pole/zero as this index. This implies that as $\psi \rightarrow 0$ we get an extra pole in the superpotential. This implies that the RS torsion behaves as

$$RS = \frac{1}{\psi} + \dots$$

where the \dots are non-singular. However since the field theory limit is obtained in the limit where the size of S goes to infinity, or equivalently the $M \rightarrow 0$ the subleading terms are suppressed by powers of Planck constant. Thus we would expect a superpotential of the form

$$W = \frac{1}{\det M} \exp \frac{-1}{g^2}$$

in perfect accord with field theoretic expectations³. In fact we can derive this more directly using the D-brane techniques. Consider the limit where we turn off all the instantons. In this case we have N_f 3-branes which fill the space. In the presence of the Euclidean threebrane we get the additional bosonic modes $\alpha^i, \tilde{\alpha}_i$ coming from open string of zero

² The Ray-Singer torsion in twisted N=1 SUSY theories has been first considered in [24].

³ Morally the Ray-Singer torsion is a holomorphic function of ψ , but as found in cases studied in [26] one expects holomorphic anomalies for this quantity. However the poles can be shown to always be holomorphic functions.

length stretched between the Euclidean 3-brane and the N_f physical 3-branes. Moreover one finds a four point interaction of fields realized by open strings between the physical 3-branes, the 7-brane and the Euclidean 3-brane, of the form

$$S = Q_i \tilde{Q}^j \alpha^i \tilde{\alpha}_j$$

To leading order this leads to the contribution to the instantons of the form

$$\int d\alpha^i d\alpha_j \exp(Q_i \tilde{Q}^j \alpha^i \tilde{\alpha}_j) = \frac{1}{\det Q_i \tilde{Q}^j}.$$

which is as expected.

3. Seiberg's $N = 1$ Duality

One of the striking aspects of the system under consideration is that there is a dual description of it in terms of a different gauge group $SU(N_f - N_c)$ discovered by Seiberg [11]. How can one understand this dual description in the present context? We will argue here that this should follow from applying T-duality to the present system, which maps 7-branes to 3-branes and 3-branes to 7-branes, by doing the analog of $R \rightarrow 1/R$ along the four dimensional space represented by S .

Let us first consider the simpler case where $S = K3$. In this case, instead of getting an $N = 1$ system we obtain an $N = 2$ system with N_f hypermultiplets in the fundamental. The situation can be described in the perturbative type IIB setup, where we have N_c parallel Dirichlet 7-branes on $R^4 \times K3$. We also have N_f Dirichlet 3-branes filling R^4 and corresponding to N_f points on $K3$. This does not mean that the net 3-brane charge is just N_f . This is because the curvature of $K3$ induces -1 unit of 3-brane charge for each 7-brane [23][27]. Since we have (N_c) 7-branes we should get a $-N_c$ contribution to the 3-brane charge. This shift in lower D-brane charge was crucial in checking the consequences of string-string duality [23][16]. Taking into account the N_f 3-branes which we have introduced we find that we have a net $N_f - N_c$ 3-brane charge. Now we apply a T-duality on $K3$ which is a total inversion of the volume of $K3$. This in particular map the 7-brane charges to 3-brane charges and vice-versa. One way to realize this is to consider $K3$ as an orbifold T^4/\mathbf{Z}_2 and apply the usual $R \rightarrow 1/R$ duality on each of the 4 circles of T^4 . We now end up with $N_f - N_c$ 7-branes which fill the dual $K3$ times the uncompactified spacetime. We now must have the net 3-brane charge of N_c . Since we

now have $N_f - N_c$ 7-branes on the dual $K3$ which induce $-N_f + N_c$ the brane charge, we must thus explicitly have N_f threebranes in addition. Thus under the T-duality we find the dual gauge group to be $SU(N_f - N_c)$ again with N_f fundamental hypermultiplets. This thus shows that the higgs branch of $N = 2$ theories with $SU(N_c)$ gauge group with N_f fundamental hypermultiplets should be the same as that of $SU(N_f - N_c)$ again with N_f fundamental hypermultiplets! This has already been noted in [28]. Note that in the above if we had used $S = T^4$ instead of $K3$ a similar consideration would have applied, except that we would have additional adjoint hypermultiplets, and that the duality would simply exchange $N_f \leftrightarrow N_c$ without a shift in N_f , because there is no lower D-brane charge induced on flat T^4 . In this case the duality is the well known duality of the exchange of instanton number with the rank of the group, and is known as the Fourier-Mukai transform [29].

Now we come to the $N = 1$ case. We will now argue that under the T-duality applied to S the rank of the dual gauge group and the number of hypermultiplets is exactly as in the $N = 2$ case discussed above where we took $S = K3$. The easiest way to see this is to consider a special case of S namely

$$S = \mathbf{P}^1 \times \mathbf{P}^1 = \frac{T^2}{\mathbf{Z}_2} \times \frac{T^2}{\mathbf{Z}_2} = \frac{T^4/\mathbf{Z}_2}{\mathbf{Z}_2} = \frac{K3}{\mathbf{Z}_2}$$

where the last \mathbf{Z}_2 acts as an inversion on one of the T^2 's. It may appear that we are breaking supersymmetry by taking the last \mathbf{Z}_2 quotient. This is not necessarily the case, because this \mathbf{Z}_2 can act on the normal direction to S in a way preserving the supersymmetry, which does not affect the geometry of S itself. The effects of the last \mathbf{Z}_2 will be at most inducing some 5-brane charges, because it leaves a number of T^2 's unchanged. This will thus not affect the 3-brane charge computed above in the context of $K3$. Thus the T-duality applied to this case should still end up giving $SU(N_f - N_c)$ as the dual magnetic gauge group with N_f hypermultiplets (q, \tilde{q}) . How about the magnetic Meson and the superpotential it has with the magnetic quarks [11]? We do not have a derivation of this from first principle because we do not know the details of how the T-duality acts on S . But with a simple assumption about how the T-duality acts, we can also recover the Meson fields in the magnetic description: Suppose that under the T-duality the N_f 3-branes that we end up with are necessarily close to each other (this can in principle arise if we consider the field theory limit in which we take the volume of S to be big). If this happens we end up getting an $SU(N_f)$ theory with 3 adjoint hypermultiplets (the $N = 4$ system on N_f coinciding

3-branes) and in addition the magnetic quarks are in the $(N_f, N_f - N_c)$ representations of $SU(N_f) \times SU(N_f - N_c)$. Note that the $SU(N_f)$ theory is thus not asymptotically free, and so there is trivial infrared dynamics associated with it. Except that out of the three adjoint fields of $SU(N_f)$ one of them couples to the magnetic quarks by a superpotential term

$$W = qM\tilde{q}.$$

This follows from the $N = 2$ structure of the interaction between the 3-brane and the 7-brane (induced from the three open string interaction representing open strings stretched between the 3-branes (M) and between the 3-brane and 7-branes (q, \tilde{q}). More precisely, one may consider the situation when the gauge coupling constant of the theory on the 3-brane is small but not zero. The effective action for the meson field can be obtained by integrating over rest of the fields on the 3-branes. To the leading approximation the effective action for the meson field consists of the free kinetic term plus above superpotential. In other words, the net effective theory in the infrared is some free theory, together with an $SU(N_f - N_c)$ gauge symmetry with N_f flavors in the fundamental representation and a neutral $N_f \times N_f$ meson field M with the classical superpotential term $W = qM\tilde{q}$, exactly as is expected in the duality proposed by Seiberg. Note that the above mechanism of an appearance of the meson field in the magnetic theory is reminiscent of how it appears in [30].

We now give some additional supporting evidence to the above picture of $N = 1$ duality. Note that under T-duality the volume of S is expected to be inverted. This may be subject to worldsheet corrections, but the general statement that increasing the volume of S should decrease the volume of the T-dual should be true. Given the relation of the volumes to the coupling constant ($V \propto \frac{1}{g^2}$) this is indeed in agreement with the fact that increasing the coupling of the electric theory should decrease the coupling of the magnetic theory and vice-versa.

4. Incorporation of Adjoint Matter

One way to incorporate adjoint matter into the above system is to allow S to have non-vanishing $h^{1,0}$ and $h^{2,0}$. In this case we expect $h^{1,0} + h^{2,0}$ adjoint hypermultiplets [3]. However there will be superpotential terms. In particular the adjoint fields coming from $h^{2,0}$ will couple to the hypermultiplet matter, just as they would in the case $S = K3$, where we would have an $N = 2$ theory. It is also conceivable that there would be superpotential terms involving the adjoint $h^{2,0}$ fields, of the type considered by Kutasov [31]. This is

possible because the zero modes of $h^{2,0}$ will have zeros along some divisor D in S and this could be a potential source for interaction. Let us note that the dimension of instanton moduli space in this case is given by (1.4)

$$\dim \mathcal{M} = 2N_f N_c - \dim(SU(N_c))(1 - h^{1,0} + h^{2,0})$$

We can interpret this formula in the gauge theory setup by noting that we have an extra contribution to the matter moduli from the adjoints coming from $h^{1,0}$ fields which do not have any superpotential associated with it. On the other hand, due to the superpotential terms associated with the $h^{2,0}$ adjoints, in the Higgs branch their expectation value is zero. Moreover their equation of motion leads to a constraint which should thus cut down the dimension of Higgs branch by $h^{2,0} \dim SU(N_c)$. This is in accord with the above formula for the dimension of instanton moduli space. It would be interesting to verify that not only the dimensions match but also the moduli spaces are identical.

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References

- [1] S. Katz, A. Klemm and C. Vafa, “Geometric Engineering of Quantum Field Theories”, hep-th/9609239.
- [2] K. Intriligator and N. Seiberg, “Lectures on supersymmetric gauge theories and electric-magnetic duality”, Nucl. Phys. Proc. Suppl. **45 BC** (1996) 1.
- [3] S. Katz and C. Vafa, “Geometric Engineering of N=1 Quantum Field Theories”, hep-th/9611090.
- [4] M. Bershadsky, K. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, “Geometric Singularities and Enhanced Gauge Symmetries”, hep-th/9605200.
- [5] S. Katz and C. Vafa, “Matter From Geometry”, hep-th/9606086.
- [6] S. Katz, private communication.
- [7] M. Bershadsky and A. Johansen, “Colliding Singularities in F-theory and Phase Transitions”, hep-th/9610111.
- [8] S. Sethi, C. Vafa and E. Witten, “Constraints on Low-Dimensional String Compactifications”, hep-th/9606122.
- [9] K. Becker and M. Becker, “M-Theory on Eight-Manifolds”, Nucl. Phys. **B 477** (1996) 155.
- [10] M. Bershadsky, V. Sadov and C. Vafa, “D-strings on D-Manifolds”, Nucl. Phys. **B 463** (1996) 398.
- [11] N. Seiberg, “Electric-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories” Nucl. Phys. **B 435** (1995) 129.
- [12] M. A. Luty and W. Taylor IV, “Varieties of vacua in classical supersymmetric gauge theories”, Phys. Rev. **D 53** (1996) 3399.
- [13] N. Seiberg, Phys. Rev. **D 49** (1994) 6857.
- [14] E. Witten, Nucl. Phys. **B460** (1996) 541.
- [15] M. Douglas, *Gauge Fields and D-branes*, hep-th/9604198.
- [16] C. Vafa, “Instantons on D-branes”, Nucl. Phys. **B463** (1996) 435.
- [17] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, “On Four-Dimensional Compactifications of F theory”, to appear.
- [18] Wei-Ping Li and Zhenbo Qin, *Stable vector bundles on algebraic surfaces*, Trans. Amer. Math. Soc. **345** (1994), no. 2, 833.
- [19] M. Maruyama, Elementary transformations in the theory of algebraic vector bundles. Algebraic geometry (La Rabida, 1981), 241–266, Lecture Notes in Math., 1961.
- [20] I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. **B 241** (1984) 493; Nucl. Phys. **B 256** (1985) 557.
- [21] K. Becker, M. Becker and A. Strominger, “Fivebranes, Membranes and Non-Perturbative String Theory”, Nucl. Phys. **B 456** (1995) 130.

- [22] E. Witten, “Non-Perturbative Superpotentials In String Theory”, Nucl.Phys. **B 474** (1996) 343.
- [23] M. Bershadsky , V. Sadov and C. Vafa, “D-Branes and Topological Field Theories”, Nucl. Phys. **B 463** (1996) 420.
- [24] A. Johansen, Int. J. Mod. Phys. **A 10** (1995) 4325.
- [25] E. Witten, private communication.
- [26] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Comm. Math. Phys **165** (1994) 311.
- [27] M. Green, J. Harvey, G. Moore, “I-Brane Inflow and Anomalous Couplings on D-Branes”, hep-th/9605033.
- [28] I. Antoniadis and B. Pioline, “Higgs branch, hyperkähler quotient and duality in SUSY N=2 Yang-Mills theories”, hep-th/9607058.
- [29] S. Mukai, Nagoya Math. J. **81** (1981) 153.
- [30] P. C. Argyres , M. R. Plesser and N. Seiberg, “The Moduli Space of N=2 SUSY QCD and Duality in N=1 SUSY QCD”, Nucl. Phys. **B 471** (1996) 159.
- [31] D. Kutasov, Phys. Lett. **B 351** (1995) 230;
D. Kutasov and A. Schwimmer, Phys. Lett. **B 354** (1995) 315;
D. Kutasov , A. Schwimmer and N. Seiberg, Nucl. Phys. **B 459** (1996) 455.